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Linear Algebra and its Applications 429 (2008) 972–980

LINEAR ALGEBRA AND ITS APPLICATIONS

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Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation

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Received 27 July 2006; accepted 28 November 2006
Available online 8 January 2007
Submitted by H. Bart

Dedicated to Professor Masahiro Nakamura on his 88th birthday with respect and affection

Abstract

We study concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation.

Keywords: Operator monotone function; Löwner–Heinz inequality

1. Introduction

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx, x) \geqslant 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function $f(t)$ on $(0, \infty)$ is said to be *operator monotone* if $f(A) \geqslant f(B)$ holds for any $A \geqslant B$.

K. Löwner [10] had established the deep theory on operator monotone functions and also he had given a definitive characterization of operator monotone functions as follows.

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0024-3795/$ - see front matter
doi:10.1016/j.laa.2006.11.023

Theorem L (K. Löwner). *A function $f:(0,\infty)$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty \frac{t}{t+s}\,\mathrm{d}m(s)$$

with $a\in\mathbb{R}$ and $b\geqslant 0$ and a positive measure m on $(0,\infty)$ such that

$$\int_0^\infty \frac{\mathrm{d}m(s)}{1+s}<+\infty.$$

Next we state the Löwner–Heinz inequality which is quite useful tool in this paper.

Theorem LH (Löwner–Heinz inequality).

(LH) t^{α} *is an operator monotone function for any $\alpha\in[0,1]$.*

Let $\alpha_j,\beta_j,\gamma_j,\ldots\in[0,1]$ for $j=1,2,\ldots,n$. Then the following (LH-1) *and* (LH-2) *are immediate consequences of* (LH).

(LH-1) $\left(\frac{1}{t^{\alpha_1}+\cdots+t^{\alpha_n}}+\frac{1}{t^{\beta_1}+\cdots+t^{\beta_n}}+\frac{1}{t^{\gamma_1}+\cdots+t^{\gamma_n}}+\cdots\right)^{-1}$ *is an operator monotone function, in particular,* $(t^{-\alpha_1}+t^{-\alpha_2}+\cdots+t^{-\alpha_n})^{-1}$ *is an operator monotone function.*

(LH-2) $(1+t^{-1})^{-\alpha_1}+(1+t^{-1})^{-\alpha_2}+\cdots+(1+t^{-1})^{-\alpha_n}$ *is an operator monotone function.*

Although (LH) of the Theorem LH was originally proved by Theorem L [10] and secondly by Heinz [6], Pedersen [11] gave an elegant proof of Theorem LH without appealing to Theorem L and also Bhatia [2, Theorem V.1.9] has given a nice different proof of Theorem LH without appealing to Theorem L (also see Bhatia [3, Theorem 4.2.1]).

In this short paper, we study concrete examples of operator monotone functions obtained by only applying Theorem LH without appealing to Theorem L and also we give an elementary proof of Theorem A [4,7] stated in §3 by only applying Theorem LH.

We state the following obvious result.

Lemma 1

$$\text{If } T\geqslant 0,\ \text{then } T^{\frac{k}{n}}-I=\left(T^{\frac{1}{n}}-I\right)\left(T^{\frac{k-1}{n}}+T^{\frac{k-2}{n}}+\cdots+T^{\frac{1}{n}}+I\right) \tag{1.1}$$

for any natural number n and k such that $1\leqslant k\leqslant n$. In particular

$$\text{if } T\geqslant 0, \text{then } T-I=\left(T^{\frac{1}{n}}-I\right)\left(T^{1-\frac{1}{n}}+T^{1-\frac{2}{n}}+\cdots+T^{\frac{1}{n}}+I\right)$$

for any natural number n.

$$\lim_{n\to\infty} n\left(T^{\frac{1}{n}}-I\right)=\log T \text{ holds for any } T>0. \tag{1.2}$$

2. Concrete examples of operator monotone functions derived from $\lim_{n\to\infty} n\left(T^{\frac{1}{n}}-I\right)=\log T$ and Löwner–Heinz inequality

Theorem 2.1

(i) $f(t)=\frac{1}{(1+t)\log\left(1+\frac{1}{t}\right)}$ *is an operator monotone function.*

(ii) $g(t)=t(1+t)\log\left(1+\frac{1}{t}\right)$ *is an operator monotone function.*

Proof. Let $A \geqslant B > 0$.

(i) We have only to show the following (2.1) on order to (i)

$$\frac{I}{(I+A)\log(I+A^{-1})} \geqslant \frac{I}{(I+B)\log(I+B^{-1})}. \tag{2.1}$$

By easy calculations, we have

$$\begin{aligned}
&\frac{I}{(I+A)n\left\{(I+A^{-1})^{\frac{1}{n}}-I\right\}} \\
&= \frac{I+A^{-1}-I}{(I+A^{-1})n\left\{(I+A^{-1})^{\frac{1}{n}}-I\right\}} \\
&= \frac{\left\{(I+A^{-1})^{\frac{1}{n}}-I\right\}\left\{(I+A^{-1})^{1-\frac{1}{n}}+(I+A^{-1})^{1-\frac{2}{n}}+\cdots+(I+A^{-1})^{\frac{1}{n}}+I\right\}}{(I+A^{-1})n\left\{(I+A^{-1})^{\frac{1}{n}}-I\right\}} \quad \text{by (1.1)} \\
&= \frac{1}{n}\left\{(I+A^{-1})^{-\frac{1}{n}}+(I+A^{-1})^{-\frac{2}{n}}+\cdots+(I+A^{-1})^{\frac{1}{n}-1}+(I+A^{-1})^{-1}\right\} \\
&\geqslant \frac{1}{n}\left\{(I+B^{-1})^{-\frac{1}{n}}+(I+B^{-1})^{-\frac{2}{n}}+\cdots+(I+B^{-1})^{\frac{1}{n}-1}+(I+B^{-1})^{-1}\right\} \quad \text{by (LH-2)} \\
&= \frac{I}{(I+B)n\left\{(I+B^{-1})^{\frac{1}{n}}-I\right\}}
\end{aligned}$$

and tending $n \to \infty$, we have (2.1) by (1.2), so the proof of (i) is complete.

(ii) We have only to show the following (2.2) in order to prove (ii)

$$A(I+A)\log(I+A^{-1}) \geqslant B(I+B)\log(I+B^{-1}). \tag{2.2}$$

By easy calculations, we have

$$\begin{aligned}
&A(I+A)n\left\{(I+A^{-1})^{\frac{1}{n}}-I\right\} \\
&= \frac{(I+A^{-1})n\left\{(I+A^{-1})^{\frac{1}{n}}-I\right\}}{(I+A^{-1}-I)(I+A^{-1}-I)} \\
&= \left(\frac{I+A^{-1}}{I+A^{-1}-I}\right)\frac{n\left\{(I+A^{-1})^{\frac{1}{n}}-I\right\}}{\left\{(I+A^{-1})^{\frac{1}{n}}-I\right\}\left\{(I+A^{-1})^{1-\frac{1}{n}}+(I+A^{-1})^{1-\frac{2}{n}}+\cdots+(I+A^{-1})^{\frac{1}{n}}+I\right\}} \\
&= \frac{n}{(I+A^{-1})^{1-\frac{1}{n}}+(I+A^{-1})^{1-\frac{2}{n}}+\cdots+(I+A^{-1})^{\frac{1}{n}}+I-(I+A^{-1})^{\frac{-1}{n}}-(I+A^{-1})^{\frac{-2}{n}}\cdots-(I+A^{-1})^{-1}} \\
&\geqslant \frac{n}{(I+B^{-1})^{1-\frac{1}{n}}+(I+B^{-1})^{1-\frac{2}{n}}+\cdots+(I+B^{-1})^{\frac{1}{n}}+I-(I+B^{-1})^{\frac{-1}{n}}-(I+B^{-1})^{\frac{-2}{n}}\cdots-(I+B^{-1})^{-1}} \\
&= B(I+B)n\left\{(I+B^{-1})^{\frac{1}{n}}-I\right\}
\end{aligned}$$

since the second equality follows by (1.1) and the inequality follows by an immediate consequence of (LH), and tending $n \to \infty$, we have (2.2) by (1.2) and the proof of (ii) is complete. $\square$

Theorem 2.2

(i) $f(t) = \frac{t-1-\log t}{\log^2 t}$ *is an operator monotone function.*

(ii) $g(t) = \frac{t\log^2 t}{t-1-\log t}$ *is an operator monotone function.*

Proof. Let $A \geqslant B > 0$.

(i) We have only to show the following (2.3) in order to prove (i)

$$\frac{A-I-\log A}{\log^2 A} \geqslant \frac{B-I-\log B}{\log^2 B}. \tag{2.3}$$

By easy calculations, we have

$$\begin{aligned}
&\frac{A-I-n(A^{\frac{1}{n}}-I)}{n\left(A^{\frac{1}{n}}-I\right)n\left(A^{\frac{1}{n}}-I\right)}\\
&\quad=\frac{\left(A^{\frac{1}{n}}-I\right)\left(A^{\frac{n-1}{n}}+A^{\frac{n-2}{n}}+\cdots+A^{\frac{1}{n}}+I\right)-n\left(A^{\frac{1}{n}}-I\right)}{n\left(A^{\frac{1}{n}}-I\right)n\left(A^{\frac{1}{n}}-I\right)}\quad\text{by (1.1)}\\
&\quad=\frac{\left(A^{\frac{n-1}{n}}+A^{\frac{n-2}{n}}+\cdots+A^{\frac{1}{n}}+I\right)-n}{n^2\left(A^{\frac{1}{n}}-I\right)}\\
&\quad=\frac{1}{n^2\left(A^{\frac{1}{n}}-I\right)}\left\{\left(A^{\frac{n-1}{n}}-I\right)+\left(A^{\frac{n-2}{n}}-I\right)+\cdots+\left(A^{\frac{1}{n}}-I\right)\right\}\\
&\quad=\frac{\left(A^{\frac{1}{n}}-I\right)}{n^2\left(A^{\frac{1}{n}}-I\right)}\left\{\left(A^{\frac{n-2}{n}}+A^{\frac{n-3}{n}}+\cdots A^{\frac{1}{n}}+I\right)\right.\\
&\qquad\left.+\left(A^{\frac{n-3}{n}}+A^{\frac{n-4}{n}}+\cdots+A^{\frac{1}{n}}+I\right)+\cdots+\left(A^{\frac{1}{n}}+I\right)+I\right\}\quad\text{by (1.1)}\\
&\quad=\frac{1}{n^2}\left\{\left(A^{\frac{n-2}{n}}+A^{\frac{n-3}{n}}+\cdots A^{\frac{1}{n}}+I\right)+\left(A^{\frac{n-3}{n}}+A^{\frac{n-4}{n}}+\cdots A^{\frac{1}{n}}+I\right)\right.\\
&\qquad\left.+\cdots+\left(A^{\frac{1}{n}}+I\right)+I\right\}\\
&\quad\geqslant\frac{1}{n^2}\left\{\left(B^{\frac{n-2}{n}}+A^{\frac{n-3}{n}}+\cdots B^{\frac{1}{n}}+I\right)+\left(B^{\frac{n-3}{n}}+B^{\frac{n-4}{n}}+\cdots B^{\frac{1}{n}}+I\right)\right.\\
&\qquad\left.+\cdots+\left(B^{\frac{1}{n}}+I\right)+I\right\}\quad\text{by (LH)}\\
&\quad=\frac{B-I-n\left(B^{\frac{1}{n}}-I\right)}{n\left(B^{\frac{1}{n}}-I\right)n\left(B^{\frac{1}{n}}-I\right)}
\end{aligned}$$

and tending $n \to \infty$, we have (2.3) by (1.2).

(ii) We have only to show the following (2.4) in order to prove (ii):

$$\frac{A\log^2 A}{A-I-\log A} \geqslant \frac{B\log^2 B}{B-I-\log B}. \tag{2.4}$$

By easy calculation, we have

$$
\begin{aligned}
&\frac{An\left(A^{\frac{1}{n}}-I\right)n\left(A^{\frac{1}{n}}-I\right)}{A-I-n\left(A^{\frac{1}{n}}-I\right)}\\
&=\frac{An\left(A^{\frac{1}{n}}-I\right)n\left(A^{\frac{1}{n}}-I\right)}{\left(A^{\frac{1}{n}}-I\right)\left(A^{1-\frac{1}{n}}+A^{1-\frac{2}{n}}+\cdots+A^{\frac{1}{n}}+I\right)-n\left(A^{\frac{1}{n}}-I\right)} \quad \text{by (1.1)}\\
&=\frac{An^2\left(A^{\frac{1}{n}}-I\right)}{\left(A^{1-\frac{1}{n}}+A^{1-\frac{2}{n}}+\cdots+A^{\frac{1}{n}}+I\right)-n}\\
&=\frac{An^2\left(A^{\frac{1}{n}}-I\right)}{\left(A^{1-\frac{1}{n}}-I\right)+\left(A^{1-\frac{2}{n}}-I\right)+\cdots+\left(A^{\frac{1}{n}}-I\right)}\\
&=\frac{An^2\left(A^{\frac{1}{n}}-I\right)}{\left(A^{\frac{1}{n}}-I\right)\left\{\left(A^{1-\frac{2}{n}}+A^{1-\frac{3}{n}}+\cdots+I\right)+\left(A^{1-\frac{3}{n}}+A^{1-\frac{4}{n}}+\cdots+I\right)+\cdots+\left(A^{\frac{1}{n}}+I\right)+I\right\}}\\
&=\frac{An^2}{\left(A^{1-\frac{2}{n}}+A^{1-\frac{3}{n}}+\cdots+I\right)+\left(A^{1-\frac{3}{n}}+A^{1-\frac{4}{n}}+\cdots+I\right)+\cdots+\left(A^{\frac{1}{n}}+I\right)+I}\\
&=\frac{n^2}{\left(A^{-\frac{2}{n}}+A^{-\frac{3}{n}}+\cdots+A^{-1}\right)+\left(A^{-\frac{3}{n}}+A^{1-\frac{4}{n}}+\cdots+A^{-1}\right)+\cdots+\left(A^{\frac{1}{n}-1}+A^{-1}\right)+A^{-1}}\\
&\geqslant\frac{n^2}{\left(B^{-\frac{2}{n}}+B^{-\frac{3}{n}}+\cdots+B^{-1}\right)+\left(B^{-\frac{3}{n}}+B^{1-\frac{4}{n}}+\cdots+B^{-1}\right)+\cdots+\left(B^{\frac{1}{n}-1}+B^{-1}\right)+B^{-1}}\\
&=\frac{Bn\left(B^{\frac{1}{n}}-I\right)n\left(B^{\frac{1}{n}}-I\right)}{B-I-n\left(B^{\frac{1}{n}}-I\right)}
\end{aligned}
$$

since the inequality follows by (LH-1) and tending $n \to \infty$, we have (2.4) and the proof is complete. □

Remark 2.1. Let $f(t)$ be a continuous function $(0, \infty) \to (0, \infty)$. It is known that $f(t)$ is an operator monotone if and only if $g(t) = \frac{t}{f(t)} = f^*(t)$ is also an operator monotone (for example, [5,8,9]) so that (i) is equivalent to (ii) in Theorems 2.1 and 2.2, here we state two direct and elementary proofs of (i) and (ii) respectively. Although several examples of operator monotone functions are shown in [9], we state an elementary method to construct concrete examples of operator monotone functions by only applying Theorem LH without appealing to Theorem L.

By using similar methods as the proofs of Theorems 2.1 and 2.2, we state the following two theorems.

Theorem 2.3. $f(t) = \frac{t(t+2)}{(t+1)^2}\log(t+2)$ *is an operator monotone function.*

Proof. Let $A \geqslant B > 0$. We have only to prove the following (2.5):

$$\frac{A(A+2)}{(A+1)^2}\log(A+2) \geqslant \frac{B(B+2)}{(B+1)^2}\log(B+2). \tag{2.5}$$

By repeating of elementary calculations, we have

$$
\begin{aligned}
&\frac{A(A+2)}{(A+1)^2} n\left\{(A+2)^{\frac{1}{n}}-I\right\} \\
&= \frac{A(A+2)}{(A+1)(A+2-1)} n\left\{(A+2)^{\frac{1}{n}}-I\right\} \\
&= \left(I-\frac{I}{A+2-I}\right)\left(\frac{n}{(A+2)^{-\frac{1}{n}}+(A+2)^{-\frac{2}{n}}+\cdots+(A+2)^{\frac{1}{n}-1}+(A+2)^{-1}}\right) \quad \text{by (1.1)} \\
&= \frac{n}{(A+2)^{-\frac{1}{n}}+(A+2)^{-\frac{2}{n}}+\cdots+(A+2)^{\frac{1}{n}-1}+(A+2)^{-1}} \\
&\quad -\left(\frac{n}{(A+2)^{1-\frac{1}{n}}+(A+2)^{1-\frac{2}{n}}+\cdots+(A+2)^{\frac{1}{n}}+I-(A+2)^{\frac{-1}{n}}-(A+2)^{\frac{-2}{n}}-\cdots-(A+2)^{-1}}\right) \\
&\geqslant \frac{n}{(B+2)^{-\frac{1}{n}}+(B+2)^{-\frac{2}{n}}+\cdots+(B+2)^{\frac{1}{n}-1}+(B+2)^{-1}} \\
&\quad -\left(\frac{n}{(B+2)^{1-\frac{1}{n}}+(B+2)^{1-\frac{2}{n}}+\cdots+(B+2)^{\frac{1}{n}}+I-(B+2)^{\frac{-1}{n}}-(B+2)^{\frac{-2}{n}}-\cdots-(B+2)^{-1}}\right) \\
&\quad \text{by (LH}-1\text{) and an immediate consequence of (LH)} \\
&= \frac{B(B+2)}{(B+1)^2} n\left\{(B+2)^{\frac{1}{n}}-I\right\}
\end{aligned}
$$

tending $n \to \infty$, we have (2.5) by (1.2), so the proof is complete. □

Theorem 2.4. $f(t) = \frac{t(t+1)}{(t+2)\log(t+2)}$ *is an operator monotone function.*

Proof. Let $A \geqslant B > 0$. We have only to prove the following (2.6)

$$\frac{A(A+I)}{(A+2)\log(A+2)} \geqslant \frac{B(B+I)}{(B+2)\log(B+2)}. \tag{2.6}$$

By repeating of elementary calculations, we have

$$
\begin{aligned}
&\frac{A(A+I)}{(A+2)n\left\{(A+2)^{\frac{1}{n}}-I\right\}} \\
&= \frac{A(A+2-I)}{(A+2)n\left\{(A+2)^{\frac{1}{n}}-I\right\}} \\
&= \frac{A}{n}\left((A+2)^{\frac{-1}{n}}+(A+2)^{\frac{-2}{n}}+\cdots(A+2)^{-1}\right) \quad \text{by (1.1)} \\
&= \frac{(A+2-2)}{n}\left((A+2)^{\frac{-1}{n}}+(A+2)^{\frac{-2}{n}}+\cdots(A+2)^{-1}\right) \\
&= \frac{1}{n}\Big((A+2)^{1-\frac{1}{n}}+(A+2)^{1-\frac{2}{n}}+\cdots(A+2)^{\frac{1}{n}}+I \\
&\quad -2(A+2)^{\frac{-1}{n}}-2(A+2)^{\frac{-2}{n}}-2(A+2)^{-1}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant \frac{1}{n}\Big((B+2)^{1-\frac{1}{n}} + (B+2)^{1-\frac{2}{n}} + \cdots (B+2)^{\frac{1}{n}} + I \\
&\quad -2(B+2)^{\frac{-1}{n}} - 2(B+2)^{\frac{-2}{n}} - 2(B+2)^{-1}\Big) \\
&= \frac{B(B+I)}{(B+2)n\left\{(B+2)^{\frac{1}{n}} - I\right\}}
\end{aligned}
$$

since the inequality follows by immediate consequences of (LH) and tending $n \to \infty$, we have (2.6) by (1.2), so the proof is complete. □

Corollary 2.5

(i) $f(t) = \frac{(t^2-1)\log(1+t)}{t^2}$ *is an operator monotone function.*
(ii) $g(t) = \frac{t(t-1)}{(t+1)\log(1+t)}$ *is an operator monotone function.*

Proof. We have only to replace t by $t-1$ in Theorems 2.3 and 2.4 respectively. □

3. Elementary proof of the result that $f_p(t) = \frac{p-1}{p}\left(\frac{t^p-1}{t^{p-1}-1}\right)$ is operator monotone for $-1 \leqslant p \leqslant 2$ by only using Löwner–Heinz inequality

The following Theorem A is shown in [4] by using Bendant–Sharman theorem [1] and also Theorem A is shown in [7] by using Pick functions closely related to Theorem L, and we shall give an elementary proof of Theorem A by only applying Löwner–Heinz inequality without appealing to Theorem L.

Theorem A. $f_p(t) = \frac{p-1}{p}\left(\frac{t^p-1}{t^{p-1}-1}\right)$ *is an operator monotone function for* $-1 \leqslant p \leqslant 2$.

$f_p(t)$ in Theorem A contains several useful means, for example,

$$f_2(t) = \frac{t+1}{2} \text{ (arithmetic mean)},$$
$$f_1(t) = \frac{t-1}{\log t} \text{ (logarithmic mean)},$$
$$f_{\frac{1}{2}}(t) = \sqrt{t} \text{ (geometric mean)}$$

and

$$f_{-1}(t) = \frac{2}{t^{-1}+1} \text{ (harmonic mean)}.$$

At first we state the following fundamental result.

Proposition 3.1. $g_p(t) = \frac{t-1}{t^p-1}$ *is an operator monotone function for* $p \in (0, 1]$.

Proof. We have only to prove the result for $p = \frac{k}{n}$ for natural numbers n and k such that $n \geqslant k \geqslant 1$ by continuity of an operator

$$\begin{aligned}
g_p(t) &= \frac{t-1}{t^{\frac{k}{n}}-1} = \frac{\left(t^{\frac{1}{n}}-1\right)\left(t^{\frac{n-1}{n}}+t^{\frac{n-2}{n}}+\cdots+t^{\frac{k}{n}}+t^{\frac{k-1}{n}}+\cdots t^{\frac{1}{n}}+1\right)}{\left(t^{\frac{1}{n}}-1\right)\left(t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+\cdots+t^{\frac{1}{n}}+1\right)} \\
&= 1+\frac{t^{\frac{n-1}{n}}+t^{\frac{n-2}{n}}+\cdots+t^{\frac{k}{n}}}{t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+\cdots+t^{\frac{1}{n}}+1} \\
&= 1+\frac{1}{t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+\cdots+t^{\frac{1}{n}}+1}\sum_{l=1}^{n-k} t^{\frac{n-l}{n}} \\
&= 1+\left(t^{\frac{-(n-k)}{n}}+t^{\frac{-(n-k+1)}{n}}+\cdots+t^{\frac{-(n-1)}{n}}\right)^{-1} \\
&\quad +\left(t^{\frac{-(n-k-1)}{n}}+t^{\frac{-(n-k)}{n}}+\cdots+t^{\frac{-(n-2)}{n}}\right)^{-1} \\
&\quad +\cdots+\left(t^{\frac{-1}{n}}+t^{\frac{-2}{n}}+\cdots+t^{\frac{-k}{n}}\right)^{-1} \qquad (*)
\end{aligned}$$

so that $g_p(t)$ is an operator monotone function by (LH-1). □

Proof of Theorem A. It suffices to prove the result for all rational numbers $p \in [-1, 2]$ by continuity of an operator.

(i) in case $1 < p \leqslant 2$. Then we have
$$f_p(t) = \frac{p-1}{p}\left(\frac{t^p-1}{t^{p-1}-1}\right) = \frac{p-1}{p}\left(t+\frac{t-1}{t^{p-1}-1}\right)$$
and $f_p(t)$ is an operator monotone function by Proposition 3.1 since $p-1 \in (0, 1]$.

(ii) in case $0 < p < 1$. Then we have
$$f_p(t) = \frac{p-1}{p}\left(\frac{t^p-1}{t^{p-1}-1}\right) = \frac{p-1}{p}+\frac{1-p}{p}\left(\frac{t-1}{t^{1-p}-1}\right)$$
and $f_p(t)$ is an operator monotone function by Proposition 3.1 since $1-p \in (0, 1]$.

(iii) in case $-1 \leqslant p < 0$.
$$f_p(t) = \frac{p-1}{p}\left(\frac{t^p-1}{t^{p-1}-1}\right) = \frac{p-1}{p}\left(\frac{(t^{|p|}-1)t}{t^{|p|+1}-1}\right)$$
and we have only to prove that $h_p(t) = \frac{(t^{|p|}-1)t}{t^{|p|+1}-1}$ is operator monotone since $\frac{p-1}{p} > 0$. Put $|p| = \frac{k}{n}$ for natural numbers n and k such that $n \geqslant k \geqslant 1$. Then

$$\begin{aligned}
h_p(t) &= \frac{(t^{|p|}-1)t}{t^{|p|+1}-1} = \frac{t}{t+\frac{t-1}{t^{|p|}-1}} \\
&= \frac{t}{t+1+\frac{t^{\frac{n-1}{n}}+t^{\frac{n-2}{n}}+\cdots+t^{\frac{k}{n}}}{t^{\frac{k-1}{n}}+t^{\frac{k-2}{n}}+\cdots+t^{\frac{1}{n}}+1}} \quad \text{by } (*) \\
&= \left(1+t^{-1}+\frac{1}{t^{\frac{k}{n}}+\cdots+t^{\frac{1}{n}}}+\frac{1}{t^{\frac{k+1}{n}}+\cdots+t^{\frac{2}{n}}}\cdots+\frac{1}{t^{\frac{n-1}{n}}+\cdots+t^{\frac{n-k}{n}}}\right)^{-1}
\end{aligned}$$

and $h_p(t)$ is an operator monotone by (LH-1).

(iv) in case $p=1$. $\frac{t-1}{n\left(t^{\frac{1}{n}}-1\right)}=\frac{1}{n}\left(t^{1-\frac{1}{n}}+t^{1-\frac{2}{n}}+\cdots+t^{\frac{1}{n}}+1\right)$, so that $f_1(t)=\lim_{p\to 1} f_p(t)=\frac{t-1}{\log t}$ is operator monotone by the same way as in §2.

(v) in case $p=0$. $\frac{n\left(t^{\frac{1}{n}}-1\right)t}{t-1}=\frac{n}{t^{\frac{-1}{n}}+t^{\frac{-2}{n}}+\cdots+t^{-1}}$, so that $f_0(t)=\lim_{p\to 0} f_p(t)=\frac{t\log t}{t-1}$ is operator monotone by the same way as in §2.

Whence the proof of Theorem A is complete by (i)–(v). □

Remark 3.1. Since $\frac{t-1}{t^{|p|}-1}$ is operator monotone for $|p|\in[0,1]$ by Proposition 3.1, so is $t+\frac{t-1}{t^{|p|}-1}$, consequently it easily turns out that $h_p(t)=\frac{t}{t+\frac{t-1}{t^{|p|}-1}}=\left(t+\frac{t-1}{t^{|p|}-1}\right)^*$ is operator monotone since as stated in Remark 2.1 and we give a direct proof of operator monotonicity of $h_p(t)$ without appealing to this dual function theorem.

Moreover we remark that $f_{\frac{1}{2}-d}(t)$ and $f_{\frac{1}{2}+d}(t)$ are both operator monotone for $0\leqslant d\leqslant\frac{3}{2}$ by Theorem A and it is easily verified that $f_{\frac{1}{2}-d}(t)=\frac{t}{f_{\frac{1}{2}+d}(t)}$ holds.

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